The Divergence of High- and Low-Frequency Estimation: Causes and Consequences

William Kinlaw, Mark Kritzman, and David Turkington
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WILLIAM KINLAW, MARK KRITZMAN, AND DAVID TURKINGTON

During the period beginning in January 1990 and ending in December 2013, U.S. stocks and emerging markets stocks had similar annualized returns: 9.5% and 9.3%, respectively. Moreover, their monthly returns were 69% correlated throughout this period, and the volatility of their monthly return differences, tracking error, was 4.9% in annualized units. Therefore, one should not expect that their cumulative returns would have diverged very significantly within this period. Yet, during the three years ending in 2007, emerging markets stocks outperformed U.S. stocks by 121%, and during the three years ending in 2013, emerging markets stocks underperformed U.S. stocks by 62%. These results call into question the utility of relying on high-frequency risk estimates for constructing long-horizon portfolios.

We analyze the causes and consequences of the divergence of high- and low-frequency estimation, and we present a framework for constructing portfolios that balance short- and long-horizon optimality.

EVIDENCE OF THE DIVERGENCE OF HIGH- AND LOW-FREQUENCY ESTIMATION

Exhibit 1 shows the distribution of the triennial relative returns of U.S. and emerging markets stocks implied by monthly estimates of standard deviation and correlation, assuming that both assets’ returns are independent and identically distributed. It also shows the distribution of the triennial relative returns that actually occurred within the same sample period. This picture leaves little doubt that estimates of standard deviation and correlation from monthly returns provide little if any guidance for estimating the distribution of triennial relative returns, at least in the case of U.S. and emerging markets stocks. Whereas their correlation measured by monthly returns is 69%, the correlation of their triennial returns turns out to be only 4%!

This problem is not unique to U.S. and emerging markets stocks. Exhibit 2 presents data on a variety of asset pairs that have exhibited significant dispersion between the distributions of triennial relative returns implied by monthly returns and actual triennial relative returns. It also shows the extent to which this dispersion is caused by various features of monthly returns.

We use the term excess dispersion to refer to the fraction of the empirical distribution that falls outside the one-standard-deviation tails of the distribution implied by estimation from monthly returns. The column labeled IID shows that there is no excess dispersion between the implied distribution and the empirical distribution as long as both assets’ returns are serially independent and identi-
Exhibit 1
Excess Dispersion of U.S. and Emerging Markets Relative Returns

January 1990–December 2013

Exhibit 2 offers persuasive evidence that estimation based on monthly returns does not reliably translate to asset behavior over longer horizons. It also reveals that most of the excess dispersion arises from a combination of non-zero auto-correlations and lagged cross-correlations and not from the non-normality of monthly returns. Appendix A shows excess dispersion for a wider set of asset pairs.

The economic intuition of the divergence of high- and low-frequency estimation may differ depending on the asset pair in question; however, there are several plausible reasons to expect volatility and correlation to differ in the short term compared to the long term. For many assets, changes in the discount rate dominate shorter-term fluctuations in price, whereas changes in cash flows dominate longer-term price movements. Poterba and Summers [1988] suggest that time-varying required returns and “slowly decaying price fads” might explain strong serial correlation in U.S. stock returns, which they detect using data from 1871 through 1986. Gilbert et al. [2013] find that the CAPM beta of stocks is a more dominant factor at low frequencies than high frequencies. Ang et al. [2014] document multi-month momentum and multi-year mean reversion for a variety of asset classes in institutional portfolios. Also, lagged pricing due to appraisal valuation in illiquid asset benchmarks can cause substantial serial correlation in observed returns, as noted by Kinlaw et al. [2013]. In the next section, we analyze the mathematics of excess dispersion that apply to any asset pair.
THE DIVERGENCE OF HIGH- AND LOW-FREQUENCY ESTIMATION: CAUSES AND CONSEQUENCES

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MATHEMATICS OF EXCESS DISPERSION

Most financial analysts assume implicitly that standard deviations scale with the square root of time and that correlations are constant across the time intervals over which they are measured. However, these properties hold only if asset returns are independently distributed across time, which means that auto-correlations and lagged cross-correlations are zero. As we have just seen, this is hardly the case.

The following example, which is contrived from 20 return observations of two hypothetical assets, illustrates that it is possible for auto-correlations and lagged cross-correlations to shift a positive monthly correlation to a negative bimonthly correlation.8

\[
\begin{align*}
\text{Monthly correlation} & = 0.50 \\
\text{Correlation of first asset with lag of second asset} & = -0.20 \\
\text{Correlation of second asset with lag of first asset} & = -1.00 \\
\text{Auto-correlation of first asset} & = -0.50 \\
\text{Auto-correlation of second asset} & = -0.50 \\
\text{Bimonthly correlation} & = -0.20
\end{align*}
\]

Exhibit 3 shows the cumulative returns of these hypothetical assets. It reveals that even though these assets’ monthly returns are 50% correlated, their values never cross. Instead, they diverge significantly over the full period. This result could prevail even if these assets have the same long-term mean return.

We now present the mathematics that relates high-frequency estimation of mean, standard deviation, and correlation to their low-frequency values. We assume throughout this section that the instantaneous rates of return for all assets are normally distributed with stationary means and variances.

We first define the discrete return of an asset \( X \) over the holding period \( t - 1 \) to \( t \) as the percentage change in the price of \( X \):

\[
r_t = \frac{P_t - P_{t-1}}{P_{t-1}}
\]

The cumulative multi-period return equals the cumulative product of the quantity, one plus the single-period returns, minus one.

We define the continuously compounded return of \( X \) as the logarithm of the quantity, one plus the discrete
return. For ease of notation going forward, we denote the continuously compounded return of asset \( X \) with a lower-case \( x \):

\[
x_i = \ln \left( \frac{P_i}{P_{i-1}} \right) = \ln(1 + r_i)
\]

(2)

The cumulative multi-period continuous return equals the cumulative sum of single-period continuous returns. The standard deviation of the cumulative continuous returns of \( x \) over \( q \) periods, \( x_t + \cdots + x_{t+q-1} \), is given by:

\[
\sigma(x_t + \cdots + x_{t+q-1}) = \sigma_x \sqrt{q + 2 \sum_{i=1}^{q-1} (q-k) \rho_{x,x_{i+1}}}
\]

(3)

where \( \sigma_x \) is the standard deviation of \( x \) measured over single-period intervals. Note that if the lagged auto-correlations of \( x \) all equal zero, the standard deviation of \( x \) will scale with the square root of the horizon, \( q \). However, to the extent the lagged auto-correlations differ from zero, extrapolating the single-period standard deviation this way to the longer-horizon standard deviation may provide a poor estimate of the actual longer-horizon standard deviation. This issue was highlighted by Lo and MacKinlay [1988], who introduced a variance ratio test to evaluate whether or not asset returns follow a random walk. Their variance ratio test compares long-horizon variance to short-horizon variance. For a long horizon, which equals \( q \) short horizons, the variance ratio is defined as:

\[
VR(q) = \frac{\sigma^2(x_t + \cdots + x_{t+q-1})}{q \sigma^2(x_t)}
\]

(4)

Lo and MacKinlay proved that the following test statistic is asymptotically normally distributed with a mean equal to zero and variance equal to one:

\[
z(q) = (VR(q) - 1) \left[ \frac{2(q-1)(q-1)}{3q} \right]^{1/2}
\]

(5)

where \( n \) is the number of observations on which the statistic is computed. Lo and MacKinlay suggest multiplying \( z(q) \) by the following quantity to reduce the bias of the estimator for small samples:

\[
\left( \frac{n-q+1}{n-q} \right) \left( \frac{1-q/n}{(n-1)} \right)
\]

(6)

This adjustment ratio will always be less than one, and it will approach one as \( n \) becomes large.

Now we introduce a second asset, \( Y \), whose continuously compounded rate of return over the period \( t - 1 \) to \( t \) is denoted \( y_t \). The correlation between the cumulative returns of \( x \) and the cumulative returns of \( y \) over \( q \) periods, is given by:

\[
\rho(x_t + \cdots + x_{t+q-1}, y_t + \cdots + y_{t+q-1}) = \frac{\rho_{x,y} + \sum_{i=1}^{q-1} (q-k) \rho_{x,y_{i+k}}}{\sqrt{q + 2 \sum_{i=1}^{q-1} (q-k) \rho_{x,y_{i+k}}}}
\]

(7)
The numerator equals the covariance of the assets taking lagged correlations into account, whereas the denominator equals the product of the assets’ standard deviations as described by Equation (3). This equation allows us to assume values for the auto-correlations of \( x \) and \( y \), as well as the lagged cross-correlations between \( x \) and \( y \), in order to compute the correlations and standard deviations that these parameters imply over longer horizons. Although particular choices for auto-correlation do not uniquely determine cross-correlations, it is important to note that choices for some of the lags do impose bounds on the possible values for other lags. See Appendix C for more detail.

We can now use the long-horizon standard deviations of \( x \) and \( y \), together with their correlation, to compute the expected standard deviation of their relative performance after \( q \) periods, which we define as the tracking error between \( x \) and \( y \):

\[
TE(x,y) = \sqrt{\sigma_x^2 + \sigma_y^2 - 2\rho\sigma_x\sigma_y}
\]  

(8)

These results for long horizons are expressed in units of continuously compounded growth. The following formulas can be used to convert the mean and standard deviation of each asset into discrete units:

\[
\mu_d = e^{\mu_c - \sigma_c^2/2} - 1
\]  

(9)

\[
\sigma_d = \sqrt{e^{2\mu_c - \sigma_c^2} - 1}
\]  

(10)

where \( \mu_c \) and \( \sigma_c \) are the mean and standard deviation of the cumulative discrete returns, and \( \mu_d \) and \( \sigma_d \) are the mean and standard deviation of cumulative continuous returns. Similarly, we compute the correlation between the cumulative discrete returns in terms of the means, standard deviations, and correlation \( \rho_c \) of cumulative continuous returns:

\[
\rho_d = \frac{e^{\mu_d - \sigma_d^2} - 1}{\sqrt{(e^{\sigma_d^2} - 1)}} \frac{\sqrt{(e^{\sigma_c^2} - 1)}}{\sqrt{(e^{\sigma_c^2} - 1)}}
\]  

(11)

Next we present the comparative statics of the divergence of high- and low-frequency estimation.

**COMPARATIVE STATICS**

The mathematical relations described above enable us to explore the effect of various lagged correlations on long-horizon dispersion. We consider two assets, \( X \) and \( Y \), which both have 7.5% annualized expected returns, 15% annualized standard deviations (assuming no auto-correlation), and a contemporaneous monthly correlation of zero. As a base case, we assume that all lagged correlations are zero. We then change one (or a few) of the lagged correlations while holding everything else constant. The panels in Exhibit 4 show the excess triennial dispersion below one standard deviation and above one standard deviation of what we would expect under the base case assumption that all lagged correlations are zero. We make the following changes to the lagged correlations.

A. Vary the strength of the correlation between \( x_t \) and \( x_{t-1} \).

B. Vary the strength of the correlation between \( x_t \) and \( x_{t-1} \) and an equal size correlation between \( x_t \) and \( x_{t-2} \) that has the opposite sign.

C. Vary the strength of the correlation between \( y_t \) and \( x_{t-1} \).

D. Vary the strength of the correlation between \( y_t \) and \( x_{t-1} \) and an equal size correlation between \( y_t \) and \( x_{t-2} \) that has the opposite sign.

This analysis leads to several general observations about excess dispersion.

- More positive auto-correlation increases the standard deviation of an asset, which increases excess dispersion (Panel A). In contrast, more negative auto-correlation decreases excess dispersion.
- More positive lagged cross-correlation decreases excess dispersion (Panel C). In contrast, more negative lagged cross-correlation increases excess dispersion.
- Shorter lags have a larger impact on excess dispersion than longer lags. Panel B shows the excess dispersion resulting from two lags that have equal size but alternating signs. For example, on the far right-hand side of Panel B, the one-period lag is equal to +1.0, and the two-period lag is equal to –1.0. The positive lag dominates the net effect.
leading to an overall increase in excess dispersion. Panel D shows an analogous result for two lagged cross-correlations that differ in sign. The intuition for this effect is that shorter lags have more opportunity than longer lags to exert their influence within a given time horizon.

- Equal size positive and negative lagged correlations do not necessarily change excess dispersion by the same amount. This can be seen from Equation (8). As the correlation increases, it decreases tracking error and thereby decreases excess dispersion at an increasing rate, due to the square root in the

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**Exhibit 4**

Comparative Statics for Lagged Correlations

Panel A: Lagged First Order x Auto-Correlation

Panel B: Lagged First Order x Auto-Correlation and Opposite Sign Lagged \( t - 2 \) Auto-Correlation

Panel C: Lagged First Order Cross-Correlation

Panel D: Lagged First Order Cross-Correlation and Opposite Sign Lagged \( t - 2 \) Cross-Correlation

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**Exhibit 5**

The Effect of Horizon on Excess Dispersion

(Lagged First-Order Cross-Correlation = –50%)
formula. Conversely, as the correlation decreases, it increases tracking error and thereby increases excess dispersion at a decreasing rate.

- Changes in the assets’ growth rates do not affect excess dispersion.
- If there are no lagged effects, changes in the investment horizon do not affect excess dispersion; it will equal zero for all horizons. If there are non-zero lagged effects, however, changes in the investment horizon do have an impact on excess dispersion. This impact is usually small. This effect can be seen in Exhibit 5 where the lagged cross-correlation equals –50%. The intuition for this effect is that contemporaneous correlations tend to dominate end-of-horizon outcomes when the horizon is short. For very long horizons, the maximum impact of lagged correlations is reflected in the end-of-horizon distribution.

**BALANCING LONG- AND SHORT-HORIZON OPTIMALITY**

If investors care only about performance over short horizons or within long horizons they could construct portfolios that reflect aversion to risk based on the covariances of monthly returns. Alternatively, if they are concerned only with performance at the conclusion of long horizons, they could compute the covariance matrix from long-horizon returns, such as three years. Or, as is the most likely case, if they care about performance over both short and long horizons, they could include separate estimates of risk; one based on a covariance matrix of monthly returns and one based on a covariance matrix of triennial returns. It is straightforward to augment the standard Markowitz [1952] optimization framework to include additional terms for aversion to long-horizon risk and long-horizon variance.13

\[
E(U) = \mu - \lambda_H \sigma_H^2 - \lambda_L \sigma_L^2
\]  

(12)

In Equation (12), \(E(U)\) is expected utility, \(\mu\) equals portfolio expected return, \(\lambda_H\) is aversion to risk estimated from high-frequency returns, \(\sigma_H^2\) is portfolio variance based on high-frequency returns, \(\lambda_L\) is aversion to low-frequency risk, and \(\sigma_L^2\) is portfolio variance based on low-frequency returns.14

It is also possible to preserve the original Markowitz format by blending the two covariance matrices in accordance with one’s relative aversion to high- and low-frequency risk.

\[
E(U) = \mu - \lambda_H \sigma_H^2
\]  

(13)

We next apply this optimization framework to a hypothetical portfolio and benchmark to analyze the portfolio’s risk of underperforming the benchmark. Exhibit 6 shows the benchmark weights and the portfolio weights. The benchmark represents the aggregate asset mix of the top 500 global investors as of December 31, 2012, as reported by the publication, Pensions and Investments. We adjust the aggregate asset mix as follows. We partition the stocks into U.S. large cap, U.S. small cap, EAFE, and EM based on market-capitalization weights. We partition bonds into global sovereigns (50%), U.S. government bonds (25%), and U.S. corporate bonds (25%). We eliminate cash, real estate, private equity, hedge funds, and other categories to focus on liquid assets in which investors could plausibly take dynamic positions.

Exhibit 7 shows the extent to which monthly estimation underestimates the actual exposure to long-horizon underperformance. The left lighter-shaded bar shows the monthly tracking error, tri-annualized. The left darker-shaded bar shows the tracking error of actual, overlapping triennial returns. The right bars show the relative value-at-risk of triennial returns at a confidence level of 95%. The right lighter-shaded bar shows the value-at-risk based on the triannualized monthly tracking error. The darker-shaded bar shows the value-at-risk drawn from actual overlapping three-year returns over the same time period. Exhibit 7 reveals that exposure to longer-term underperformance is almost twice as large as an investor would have expected from monthly returns.

**EXHIBIT 6**

Benchmark and Portfolio Weights

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Benchmark Weight</th>
<th>Portfolio Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. Large Cap</td>
<td>24.33%</td>
<td>44.33%</td>
</tr>
<tr>
<td>U.S. Small Cap</td>
<td>2.70%</td>
<td>2.70%</td>
</tr>
<tr>
<td>EAFE Equities</td>
<td>19.64%</td>
<td>9.64%</td>
</tr>
<tr>
<td>Emerging Equities</td>
<td>5.80%</td>
<td>0.80%</td>
</tr>
<tr>
<td>Global Sovereigns</td>
<td>23.76%</td>
<td>18.76%</td>
</tr>
<tr>
<td>U.S. Government Bonds</td>
<td>11.88%</td>
<td>6.88%</td>
</tr>
<tr>
<td>U.S. Corporate Bonds</td>
<td>11.88%</td>
<td>16.88%</td>
</tr>
</tbody>
</table>
Exhibit 8 shows the standard deviations and correlations of the component assets estimated from monthly returns, and Exhibit 9 shows these values based on overlapping triennial returns.

Exhibit 10 shows an iso-expected return curve. The portfolios along this curve all have the same expected return, but they have different combinations of monthly and triennial tracking error. We construct this curve by first deriving a set of implied returns from the monthly covariance matrix. We then solve for the optimal weights using Equation (12), but we initially assume no aversion to triennial risk. This first optimization gives us the extreme upper-left portfolio on the curve. We then progressively increase aversion to triennial risk, while holding expected return constant, which moves us down and right along the curve. This curve represents the available choices to an investor who cares about aversion to both short-term and long-term losses.

Exhibit 10 shows that it is possible to construct portfolios that simultaneously optimize high- and low-frequency risk in a manner that is consistent with an investor’s relative aversion to these two measures of risk. Exhibit 11 shows the weights and expected excess returns of three selected portfolios along this curve as well as their monthly and triennial tracking error.
EXHIBIT 9
Triennial Standard Deviations and Correlations

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Standard Deviation</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. Large Cap</td>
<td>42.98%</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U.S. Small Cap</td>
<td>33.49%</td>
<td>0.73</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EAFE Equities</td>
<td>35.34%</td>
<td>0.57</td>
<td>0.71</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Emerging Equities</td>
<td>62.30%</td>
<td>0.04</td>
<td>0.53</td>
<td>0.71</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Global Sovereigns</td>
<td>12.14%</td>
<td>-0.15</td>
<td>0.06</td>
<td>-0.06</td>
<td>0.26</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>U.S. Government Bonds</td>
<td>8.10%</td>
<td>-0.04</td>
<td>-0.13</td>
<td>-0.51</td>
<td>-0.29</td>
<td>0.47</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>U.S. Corporate Bonds</td>
<td>15.20%</td>
<td>0.23</td>
<td>0.42</td>
<td>-0.07</td>
<td>0.04</td>
<td>0.42</td>
<td>0.42</td>
<td>1.00</td>
</tr>
</tbody>
</table>

EXHIBIT 10
Iso-Expected Return Curve Balancing Short- and Long-Horizon Tracking Error

EXHIBIT 11
Weights, Expected Excess Return, and Tracking Error of Selected Portfolios

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>Portfolio 1 (monthly)</th>
<th>Portfolio 2 (blended)</th>
<th>Portfolio 3 (triennial)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. Large Cap</td>
<td>24.33%</td>
<td>44.33%</td>
<td>37.54%</td>
<td>33.57%</td>
</tr>
<tr>
<td>U.S. Small Cap</td>
<td>2.70%</td>
<td>2.70%</td>
<td>9.44%</td>
<td>14.66%</td>
</tr>
<tr>
<td>EAFE Equities</td>
<td>19.64%</td>
<td>9.64%</td>
<td>0.45%</td>
<td>-2.46%</td>
</tr>
<tr>
<td>Emerging Equities</td>
<td>5.80%</td>
<td>0.80%</td>
<td>8.84%</td>
<td>11.69%</td>
</tr>
<tr>
<td>Global Sovereigns</td>
<td>23.76%</td>
<td>18.76%</td>
<td>12.26%</td>
<td>8.88%</td>
</tr>
<tr>
<td>U.S. Government Bonds</td>
<td>11.88%</td>
<td>6.88%</td>
<td>19.24%</td>
<td>29.75%</td>
</tr>
<tr>
<td>U.S. Corporate Bonds</td>
<td>11.88%</td>
<td>16.88%</td>
<td>12.22%</td>
<td>3.92%</td>
</tr>
<tr>
<td>Expected Excess Return*</td>
<td>n/a</td>
<td>0.82%</td>
<td>0.82%</td>
<td>0.82%</td>
</tr>
<tr>
<td>Monthly TE</td>
<td>n/a</td>
<td>0.58%</td>
<td>0.77%</td>
<td>0.96%</td>
</tr>
<tr>
<td>Triennial TE</td>
<td>n/a</td>
<td>8.88%</td>
<td>5.58%</td>
<td>5.02%</td>
</tr>
</tbody>
</table>

* Annualized.
SUMMARY

Financial analysts are often surprised by the extent to which assets that are thought to be strongly correlated diverge over time. This divergence sometimes occurs because the sample used to estimate a set of parameters does not represent out-of-sample behavior. But this result could also occur within the same sample from which the parameters are derived. It occurs because one or both assets may have non-normal returns, or because they are auto-correlated or cross-correlated at one or more lags.

We present evidence of the excess dispersion in the distribution of relative returns for a variety of asset pairs, and we attribute this excess dispersion to its sources. We find that most excess dispersion arises from non-zero auto-correlations and lagged cross-correlations. Non-normality is not a significant contributor to excess dispersion.

We also present the mathematics that relates high-frequency estimation of mean, standard deviation, and correlation to their low-frequency values. We deploy these equations to evaluate the comparative statics of long-horizon dispersion.

Finally, we introduce a portfolio construction framework that allows investors to build portfolios that jointly account for aversion to high- and low-frequency risk. We present a simple case study based on this methodology, and we derive an iso-expected return curve. Portfolios that lie on this curve all have the same expected return, but they have different combinations of high- and low-frequency tracking error.

Our analysis reveals that high-frequency estimation does not reliably predict behavior over long horizons, even absent sampling error. Investors, therefore, are well advised to include risk estimates based on both high- and low-frequency returns when constructing portfolios for long horizons.

APPENDIX A

EXCESS DISPERSION OF TRIENNIAL RETURNS

Exhibit A1 summarizes the excess dispersion of triennial returns for all the asset classes included in Exhibit 6, plus commodities, hedge funds, real estate and private equity. The lower triangle of the matrix in Exhibit A1 shows excess dispersion, and the upper triangle of the matrix shows the correlation of triennial returns minus the correlation of monthly returns.

EXHIBIT A1
Excess Dispersion of Triennial Returns

<table>
<thead>
<tr>
<th></th>
<th>U.S. Large Cap</th>
<th>U.S. Small Cap</th>
<th>EAFE Equities</th>
<th>Emerging Markets</th>
<th>Global Sovereigns</th>
<th>U.S. Government Bonds</th>
<th>U.S. Corporate Bonds</th>
<th>Commodities</th>
<th>Hedge Funds</th>
<th>Real Estate</th>
<th>Private Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. Large Cap</td>
<td>-0.02</td>
<td>-0.06</td>
<td>-0.60</td>
<td>-0.29</td>
<td>-0.12</td>
<td>-0.11</td>
<td>-0.05</td>
<td>0.00</td>
<td>0.29</td>
<td>0.14</td>
<td></td>
</tr>
<tr>
<td>U.S. Small Cap</td>
<td>3.4%</td>
<td>0.09</td>
<td>-0.26</td>
<td>-0.12</td>
<td>-0.13</td>
<td>0.09</td>
<td>0.11</td>
<td>-0.04</td>
<td>0.32</td>
<td>-0.07</td>
<td></td>
</tr>
<tr>
<td>EAFE Equities</td>
<td>5.8%</td>
<td>-15.9%</td>
<td>-0.22</td>
<td>-0.49</td>
<td>-0.47</td>
<td>-0.39</td>
<td>-0.02</td>
<td>-0.20</td>
<td>0.44</td>
<td>0.07</td>
<td></td>
</tr>
<tr>
<td>Emerging Markets</td>
<td>27.6%</td>
<td>14.5%</td>
<td>1.1%</td>
<td>0.27</td>
<td>-0.02</td>
<td>-0.15</td>
<td>0.06</td>
<td>-0.32</td>
<td>0.21</td>
<td>-0.38</td>
<td></td>
</tr>
<tr>
<td>Global Sovereigns</td>
<td>15.3%</td>
<td>-12.8%</td>
<td>3.4%</td>
<td>-0.1%</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.03</td>
<td>0.23</td>
<td>-0.28</td>
<td>-0.32</td>
<td></td>
</tr>
<tr>
<td>U.S. Government Bonds</td>
<td>14.5%</td>
<td>-10.4%</td>
<td>-1.3%</td>
<td>3.8%</td>
<td>-4.9%</td>
<td>-0.20</td>
<td>0.02</td>
<td>0.44</td>
<td>-0.20</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>U.S. Corporate Bonds</td>
<td>14.1%</td>
<td>-15.1%</td>
<td>-0.1%</td>
<td>1.9%</td>
<td>-9.6%</td>
<td>-12.0%</td>
<td>0.06</td>
<td>-0.05</td>
<td>-0.13</td>
<td>-0.20</td>
<td></td>
</tr>
<tr>
<td>Commodities</td>
<td>3.1%</td>
<td>-15.1%</td>
<td>-4.5%</td>
<td>-4.1%</td>
<td>-7.6%</td>
<td>-3.7%</td>
<td>-5.2%</td>
<td>0.05</td>
<td>0.24</td>
<td>-0.29</td>
<td></td>
</tr>
<tr>
<td>Hedge Funds</td>
<td>14.5%</td>
<td>-15.5%</td>
<td>-2.1%</td>
<td>12.5%</td>
<td>5.4%</td>
<td>3.1%</td>
<td>4.2%</td>
<td>-3.7%</td>
<td>0.56</td>
<td>-0.08</td>
<td></td>
</tr>
<tr>
<td>Real Estate*</td>
<td>-7.2%</td>
<td>-17.7%</td>
<td>-12.4%</td>
<td>-5.4%</td>
<td>29.7%</td>
<td>22.7%</td>
<td>17.4%</td>
<td>-19.5%</td>
<td>1.9%</td>
<td>0.23</td>
<td></td>
</tr>
<tr>
<td>Private Equity*</td>
<td>-5.4%</td>
<td>5.1%</td>
<td>1.6%</td>
<td>6.9%</td>
<td>31.4%</td>
<td>26.2%</td>
<td>27.9%</td>
<td>1.6%</td>
<td>26.2%</td>
<td>-1.9%</td>
<td></td>
</tr>
</tbody>
</table>

Upper Triangle of Matrix: Triennial Correlation Minus Monthly Correlation

Lower Triangle of Matrix: Actual Excess Dispersion of Triennial Returns

* All data monthly from Jan 1990–Dec 2013 except Real Estate and Private Equity, which are quarterly from Q1 1996–Q3 2013.
Appendix B

Excess Dispersion Based on Discrete Returns (Difference of Two Log-Normally Distributed Variables)

Exhibit B1 shows the excess dispersion of the same asset pairs analyzed in Exhibit 2, but it is now estimated from the distribution of relative periodic returns, rather than from relative growth rates, as shown previously. These results suggest that the choice of discrete or continuous returns has little effect on the results. If the growth rates of the assets are normally distributed, the periodic returns of each asset will be log-normally distributed. The difference between two log-normal distributions is not itself log-normal, so we resort to simulation to estimate excess dispersion from discrete returns. We proceed as follows.

1. We estimate means, standard deviations, and correlations from the logarithms of one plus the original monthly returns, and extrapolate these to a three-year horizon using the relevant assumptions.
2. We sample 10,000 observations for the three-year continuous growth rates using the multivariate normal distribution from step 1.
3. We compute \( e^r - 1 \) for all 10,000 simulated returns \( r \), for both of the assets, to arrive at their log-normal distributions.
4. We take the difference of the asset observations from step 3 for all of the 10,000 simulated pairs to generate the simulated distribution of differences in periodic returns.
5. We estimate the relative return thresholds from the base case scenario to generate the one-standard deviation left and right tails. We compute the distance of these thresholds below and above the mean for the base case scenario and compute the percentage of observations that fall outside these thresholds for each particular case.

Exhibit B1
Simulated Excess Dispersion of Triennial Returns

<table>
<thead>
<tr>
<th>U.S. Domestic Assets</th>
<th>Excess Dispersion</th>
<th>Non-Normality</th>
<th>Auto-Correlation</th>
<th>Lagged Cross-Correlation</th>
<th>Auto- and Cross-Correlation</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hedge Funds</td>
<td>Private Equity</td>
<td>0.0%</td>
<td>0.1%</td>
<td>26.5%</td>
<td>4.0%</td>
<td>28.3%</td>
</tr>
<tr>
<td>Real Estate</td>
<td>Government Bonds</td>
<td>0.0%</td>
<td>-2.0%</td>
<td>21.3%</td>
<td>4.1%</td>
<td>23.4%</td>
</tr>
<tr>
<td>Large Cap Equity</td>
<td>Government Bonds</td>
<td>0.0%</td>
<td>1.4%</td>
<td>9.9%</td>
<td>1.0%</td>
<td>11.5%</td>
</tr>
<tr>
<td>Corporate Bonds</td>
<td>Government Bonds</td>
<td>0.0%</td>
<td>-2.3%</td>
<td>-14.9%</td>
<td>5.6%</td>
<td>-6.2%</td>
</tr>
<tr>
<td>Energy Stocks</td>
<td>Utilities Stocks</td>
<td>0.0%</td>
<td>0.4%</td>
<td>-9.7%</td>
<td>-17.2%</td>
<td>-24.3%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>International Assets</th>
<th>Excess Dispersion</th>
<th>Non-Normality</th>
<th>Auto-Correlation</th>
<th>Lagged Cross-Correlation</th>
<th>Auto- and Cross-Correlation</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. Equities</td>
<td>Emerging Equities</td>
<td>0.0%</td>
<td>-1.0%</td>
<td>-2.8%</td>
<td>20.8%</td>
<td>20.5%</td>
</tr>
<tr>
<td>Canadian Equities</td>
<td>U.S. Equities</td>
<td>0.0%</td>
<td>0.4%</td>
<td>-0.8%</td>
<td>14.8%</td>
<td>14.3%</td>
</tr>
<tr>
<td>Global Portfolio in USD</td>
<td>USD/AUD</td>
<td>0.0%</td>
<td>-0.2%</td>
<td>-2.7%</td>
<td>20.6%</td>
<td>17.8%</td>
</tr>
<tr>
<td>USD/GBP</td>
<td>USD/JPY</td>
<td>0.0%</td>
<td>-0.5%</td>
<td>-1.7%</td>
<td>12.9%</td>
<td>9.8%</td>
</tr>
<tr>
<td>German Equities</td>
<td>U.K. Equities</td>
<td>0.0%</td>
<td>-0.9%</td>
<td>-3.5%</td>
<td>-11.5%</td>
<td>-17.7%</td>
</tr>
</tbody>
</table>

Appendix C

Implied Bounds for Lagged Correlations

Taken together, a set of lagged correlations is valid only if it guarantees that volatility measured over any horizon \( q \) is non-negative, and correlations measured over any horizon \( q \) are between negative one and positive one. Therefore, we can think about the bounds imposed on the value of each successive lag in an iterative fashion. Assuming \( q - 2 \) lags (both auto-correlations and cross-correlations) have already been specified, we begin by computing the bounds for the \( q - 1 \) period auto-correlation lag of \( x \):
\[ \rho_{x,x_{t+q}} \geq -\frac{q}{2} \sum_{k=1}^{q-2} (q-k) \rho_{x,x_{k}} \]  \hspace{1cm} (C-1)

This value, together with all of the \( q - 2 \) lags, will put the following lower and upper bounds on the choice of the \( q - 1 \) period cross-correlation lags between \( x \) and \( y \):

\[
\begin{align*}
\rho_{x,x_{t+q}} + \rho_{y,y_{t+q}} & \geq -\sqrt{q + 2\sum_{k=1}^{q-2}(q-k)\rho_{x,x_{k}}} - \sqrt{q + 2\sum_{k=1}^{q-2}(q-k)\rho_{y,y_{k}}} - q\rho_{x,y} \\
\rho_{x,x_{t+q}} + \rho_{y,y_{t+q}} & \leq \sqrt{q + 2\sum_{k=1}^{q-2}(q-k)\rho_{x,x_{k}}} + \sqrt{q + 2\sum_{k=1}^{q-2}(q-k)\rho_{y,y_{k}}} - q\rho_{x,y}
\end{align*}
\]

\[ \rho_{x,x_{t+q}} + \rho_{y,y_{t+q}} \leq \sqrt{q + 2\sum_{k=1}^{q-2}(q-k)\rho_{x,x_{k}}} + \sqrt{q + 2\sum_{k=1}^{q-2}(q-k)\rho_{y,y_{k}}} - q\rho_{x,y} \]  \hspace{1cm} (C-2)

\[ \rho_{x,x_{t+q}} + \rho_{y,y_{t+q}} \leq \sqrt{q + 2\sum_{k=1}^{q-2}(q-k)\rho_{x,x_{k}}} + \sqrt{q + 2\sum_{k=1}^{q-2}(q-k)\rho_{y,y_{k}}} - q\rho_{x,y} \]  \hspace{1cm} (C-3)

ENDNOTES

The authors would like to thank Timothy Adler, Pierre Marzouk, Anya Obizhaeva, and the participants at the State Street Global Markets research conferences in London and Cambridge, MA, for helpful comments and feedback.

1These returns are based on the S&P 500 Index and MSCI Emerging Markets Index.

2We use term high frequency to refer to periods as short as a month, a week, or a day. We do not have in mind milliseconds.

3Asness et al. [2011] address this issue from the perspective of diversification. We focus on the negative consequences of excess dispersion, and we investigate the mathematical properties that explain it.

4We perform this analysis in continuous rather than discrete units. For any given asset pair, we convert the monthly returns of both assets into continuous growth rates by taking the logarithm of the quantity, one plus the relevant asset’s cumulative discrete return. In Appendix C, we show simulated results for excess dispersion of the difference (rather than the ratio) of cumulative discrete returns. The two sets of results are nearly identical. It is also worth noting that the choice of rebalancing frequency has little impact on these results. In this article, we have assumed no rebalancing, which means that both assets in a pair compound on their own and their relative returns are assessed at the end of the horizon. Results for excess dispersion are very similar if the asset values are rebalanced monthly. We do not present these results due to space constraints.

5Appendix A shows excess dispersion for a wider selection of asset pairs.

6We calculate monthly returns based on asset values from the last calendar day of each month. For assets where daily data is available, we compared these results to another set of results calculated from monthly returns measured on the 15th day of each month. The results were very similar, but there is a slight bias for the 15th-of-the-month data to exhibit less positive (or more negative) excess dispersion compared to the end-of-the-month data. In our tests, the extent of this bias was around 2% in excess dispersion units. This finding might be explained by the widespread practice of month-end portfolio rebalancing, which would have a dampening effect on the volatility of month-end returns and lead to higher excess dispersion of actual returns over long horizons.

7Throughout this article, we measure actual excess dispersion using a rolling window to capture all long-horizon returns that occurred throughout the sample. Each rolling observation is given equal weight. One possible disadvantage of this assumption is that we under-represent data points at the very beginning and end of the sample, because they are contained in fewer rolling windows than data points in the middle of the sample. This effect should be minor given that our data sample is reasonably long. To get around this issue, one could sample long-horizon blocks by splicing the beginning of the data set on to the end of the data set. We choose
not to use this technique here because it will also introduce a spurious time series relationship where the data is spliced.

Equation (9) shows the details of this transformation.

This is true even when considering continuously compounded returns for both the single-period and multi-period horizons. Comparing discrete returns to continuous returns is a separate issue, which we also address.

Kritzman [2007] discusses the relation between continuous and discrete returns, and provides formulas for converting mean and standard deviation.

The discrete returns are log-normally distributed. The log-normal distribution is not symmetrical and therefore cannot be described by just its mean and standard deviation. The conversion of mean and standard deviation into discrete units provides only an approximation to the actual expected log-normal distribution of compounded discrete returns at the end of the horizon. Blay and Markowitz [2013] provide a thorough discussion of this issue.

Although many investors claim to care only about long-horizon results, it is naïve to believe that they are indifferent to large drawdowns along the way.

Chow et al. [1999] deploy the same methodology to construct portfolios for which the investor has different degrees of aversion to risk during turbulent regimes and to risk during quiet regimes.

Mathematically, it is fine to express expected return, high-frequency variance and low-frequency variance in different time units; however, the risk-aversion coefficients will not be directly comparable. Aesthetically, it may be more pleasing to express these terms in low-frequency units, in which case the risk-aversion coefficients will be of the same scale. To convert the high-frequency variance into units of low-frequency variance, we simply multiply the high-frequency variance by the number of high-frequency intervals contained in one low-frequency interval. This adjustment ignores lagged effects, which is appropriate here because our goal is to characterize high-frequency variance, which by definition does not depend on multi-period relationships.

In these optimizations variance refers to the variance of relative returns.

The utility term associated with the monthly covariance matrix in this optimization implicitly assumes that the portfolio is rebalanced each month, whereas the triennial term assumes it is rebalanced triennially. When we combine these terms, we can think of the monthly term as an approximation of short-term risk.

There are several remedies to address estimation error arising from the use of one sample to project another sample, including Bayesian adjustment, shrinkage, and resampling.

However, this type of estimation error is unrelated to errors arising from the use of high-frequency returns to project low-frequency behavior within the same sample.

REFERENCES


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