

The Journal of Portfolio Management

VOLUME 39, NUMBER 2

JPM.pm-research.com

WINTER 2013

Liquidity and Portfolio Choice: *A Unified Approach*

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Investors have long recognized the importance of liquidity but have struggled to determine how to account for it when forming portfolios. We propose that investors treat liquidity as a shadow allocation to a portfolio. If investors deploy liquidity to raise a portfolio's expected utility beyond its original expected utility, we attach a shadow asset to tradable assets. If instead investors deploy liquidity to prevent a portfolio's expected utility from falling, we attach a shadow liability to non-tradable assets. This approach improves on other methods of incorporating liquidity into portfolio choice in four fundamental ways.

First, it mirrors what actually occurs within a portfolio. Second, it maps units of liquidity onto units of expected return and risk, so that investors can analyze liquidity in the same context as other portfolio decisions. Third, it enables investors to address absolute illiquidity and partial illiquidity within a single, unifying framework. Fourth, it recognizes that liquidity serves not only to meet demands for capital, but also lets investors exploit opportunities, revealing that investors bear an illiquidity cost to the extent that any portion of a portfolio is immobile.

RELATED LITERATURE

Researchers have considered the topic of illiquidity from a variety of perspectives. Takahashi and Alexander [2002] developed

a financial model to project illiquid assets' future values and cash flows. Perhaps the approach most similar to ours is that of Lo, Petrov, and Wierzbicki [2003]. They introduced liquidity as a constraint or additional dimension in the mean–variance optimization process. Their approach, however, does not map liquidity onto units of return and risk, so investors must treat liquidity as a separate portfolio feature. Our approach, by contrast, unifies liquidity, return, and risk.

Getmansky, Lo, and Makarov [2004] investigated illiquidity and serial correlation of hedge fund returns. Hill [2009] argued that long option positions provide a natural hedge for liquidity risk. Ang, Papanikolaou, and Westerfield [2010] investigated how trading restrictions affect portfolio choice; they showed that illiquidity increases risk aversion and distorts the allocation of liquid and illiquid assets. Anson [2010] introduced a framework for measuring liquidity risk across asset classes. Golts and Kritzman [2010] proposed that investors consider purchasing liquidity options to meet unscheduled capital calls; they described how to structure and price these options. Hu, Pan, and Wang [2010] measured illiquidity's time variation from the deviations of observed yields relative to a fitted yield curve. Beachkofski et al. [2011] simulated portfolio performance with restrictions on trading; they measured the cost of illiquidity as the certainty equivalent reduction of expected

utility. Most recently, Kyle and Obizhaeva [2011] tested the intuition that the size and costs of transferring risk in “business time” is constant across assets and time. In a companion paper, Kyle and Obizhaeva [2011] developed formulas for estimating impact and spread costs.

BENEFITS OF LIQUIDITY

In order to treat liquidity as a shadow allocation to a portfolio, we must estimate its expected return and risk, which in turn requires us to evaluate how investors benefit from liquidity. There are several obvious benefits: the ability to exercise market timing, rebalance a portfolio, meet capital calls, reallocate part of a portfolio to newly discovered opportunities, exit from unproductive investments, and respond to shifts in risk tolerance. Some of this trading improves a portfolio's expected utility and thus constitutes a shadow asset which is attached to tradable assets. Some trading, however, occurs to prevent a decrease in expected utility, in which case we attach a shadow liability to assets that are not tradable. Below we distinguish whether these activities constitute a shadow asset or a shadow liability.

Market Timing

Some investors are skilled at anticipating the relative performance of asset classes or risk factors. We attach the expected return and risk of a market timing strategy as a shadow asset to the liquid portion of the portfolio, because this trading improves expected utility beyond the portfolio's initial expected utility.

Rebalancing

Investors choose portfolios they believe are optimal, given their views and attitudes about expected return and risk. Once they establish an optimal portfolio, however, price changes cause the portfolio's actual weights to drift away from the optimal targets, making the portfolio suboptimal. If the portfolio contains only liquid assets, investors can restore the optimal weights easily, though not without cost.¹ To the extent some portion of the portfolio is allocated to illiquid assets, investors cannot implement the full solution, and the portfolio remains sub-optimal. In this case, we attach a shadow liability to illiquid assets, because their immo-

bility reduces the portfolio's expected utility. The cost of this shadow liability equals the difference between the certainty equivalent of the suboptimal portfolio and the certainty equivalent of the optimal portfolio.²

Capital Calls

Investors must periodically liquidate a portion of their portfolios to meet capital calls. Pension funds, for example, may need to make unanticipated benefit payments. Many endowment funds and foundations commit to private equity and real estate funds, which demand capital sporadically, as investment opportunities arise. Private investors must occasionally replace lost income to meet their consumption demands. These liquidations can drive a portfolio away from its optimal mix. To the extent part of the portfolio is allocated to illiquid assets, the investor may not be able to restore full optimality. As with rebalancing, we measure this cost in certainty equivalent units and attach it to the illiquid assets as a shadow liability.

In rare instances, investors may be unable to liquidate a sufficient fraction of their portfolios to meet capital calls, requiring them to borrow. In these instances, we attach another shadow liability to the illiquid assets, to reflect the cost and uncertainty of borrowing.

New Opportunities

Investors may discover new managers, new strategies, or better ways to reconfigure existing portfolios. In these circumstances, we attach a shadow asset to the portfolio, to capture the improvement expected from reconfiguring it. Alternatively, investors may wish to exit existing positions they no longer expect to perform as originally predicted. In these situations, we attach a shadow liability to illiquid assets that cannot be removed from the portfolio.

Shifting Risk Tolerance

Investors may become more or less averse to risk as their circumstances change. In this case, we attach a shadow liability to the illiquid assets, because they limit the extent to which investors can respond to their shifting risk tolerances. As with portfolio rebalancing, we measure this cost in certainty-equivalent units.

ABSOLUTE VERSUS PARTIAL ILLIQUIDITY

In our discussion thus far, we have implicitly treated liquidity and illiquidity as binary attributes. We should, instead, distinguish between absolute illiquidity and partial illiquidity. Absolute illiquidity refers to situations in which the investor is contractually proscribed from trading the asset or the cost of trading is prohibitively expensive. In these cases, we cannot attach a shadow asset. We recognize that many liquid assets are at least partially illiquid, in the sense that they are costly to trade or can be traded only with delay. Costs and delays vary by asset and through time. We attach a shadow asset to partially illiquid assets, but we reduce its expected return by the cost of trading or by the extent to which delays limit the opportunity to benefit from trading. We thus neatly capture these varying degrees of illiquidity within a single, unifying framework.

ANALYTICAL CONSTRUCT

We introduce our analytical construct by using mean–variance analysis to solve for the optimal allocation to liquid equity and liquid bonds, first without considering the effect of liquidity.³ We identify the optimal weights by maximizing expected utility:⁴

$$E(U) = r_e w_e + r_b w_b - \lambda(\sigma_e^2 w_e^2 + \sigma_b^2 w_b^2 + 2\rho_{b,e} \sigma_e \sigma_b w_e w_b) \quad (1)$$

where

- $E(U)$ = expected utility
- r_e = expected equity return
- r_b = expected bond return
- σ_e = equity standard deviation
- σ_b = bond standard deviation
- w_e = equity weight
- w_b = bond weight
- λ = coefficient of risk aversion
- $\rho_{b,e}$ = correlation of equity and bonds

The weights that equate the marginal utilities of equities and bonds, as shown below, are those that are optimal.⁵

$$\frac{\partial U}{\partial w_e} = r_e - \lambda(2\sigma_e^2 w_e + 2\rho_{b,e} \sigma_e \sigma_b w_b) \quad (2)$$

$$\frac{\partial U}{\partial w_b} = r_b - \lambda(2\sigma_b^2 w_b + 2\rho_{b,e} \sigma_e \sigma_b w_e) \quad (3)$$

Next we substitute illiquid equity for liquid equity, but first we make two adjustments. We correct for the downward bias in the illiquid asset's observed standard deviation, which arises from the effect of performance fees. For a single fund that accounts for performance fees annually, the illiquid asset's net returns can be converted to gross returns:

$$r_n = r_g - b - \max(0, p * (r_g - b)) \quad (4)$$

$$r_g = \begin{cases} r_n + b & \text{for } r_n < 0 \\ \frac{r_n}{1-p} + b & \text{for } r_n \geq 0 \end{cases} \quad (5)$$

where

- r_n = return net of fees
- r_g = return gross of fees
- b = base fee
- p = performance fee

In practice, we perform a simulation to estimate the volatility dampening effect of performance fees when fee accrual accounting is used. On average, the true standard deviation is approximately 1.09 times larger than the standard deviation estimated from monthly net returns from funds using fee accrual.⁶

The second adjustment de-smoothes illiquid equity returns, to offset the reduced observed standard deviation introduced by appraisals and fair-value pricing.⁷ We estimate a first-order autoregressive model using least squares. We specify the model as

$$r_t = A_0 + A_1 * r_{t-1} + \varepsilon \quad (6)$$

To de-smooth the time series, we compute

$$r'_t = \frac{r_t - A_1 * r_{t-1}}{1 - A_1} \quad (7)$$

where

- r'_t = de-smoothed return observation at time t
- r_t = return observation at time t
- A_0 = intercept
- A_1 = regression coefficient
- ε = error term

Following these two adjustments, we switch illiquid equity for liquid equity. Then we attach the shadow asset to the bond portion of the portfolio and re-state the expected return, standard deviation, and correlation of bonds to account for the presence of the shadow asset, as shown in equations (8), (9), and (10).⁸

$$r_{bl} = r_b + r_l \quad (8)$$

$$\sigma_{bl}^2 = \sigma_b^2 + \sigma_l^2 \quad (9)$$

$$\rho_{bl,e} = \frac{\rho_{b,e} * \sigma_b}{\sigma_{bl}} \quad (10)$$

where

r_{bl} = expected return of bonds with shadow liquidity asset

r_l = expected return of shadow liquidity asset

σ_{bl} = standard deviation of bonds with shadow liquidity asset

σ_l = standard deviation of shadow liquidity asset

$\rho_{bl,e}$ = correlation of bonds (with shadow liquidity asset) and equity

Exhibit 1 presents a simple numerical illustration of our analytical construct. It shows how the required return for equities changes as we switch from liquid equities to illiquid equities and then, step by step, adjust for the effects of performance fees, smoothing, and inclusion of the shadow asset.⁹

Column 1 shows that it is optimal to split the portfolio equally between liquid equity and liquid bonds, given the indicated assumptions for their expected returns, standard deviations, and correlation, and assuming that the investor's risk aversion coefficient equals 1. Notice that their marginal utilities are equal, demonstrating that we cannot improve expected utility by altering these weights. At this point we have not yet accounted for the expected return and risk of the shadow asset.

In column 2, we substitute illiquid equity for liquid equity and solve for the return required to produce the same weight, given the illiquid equity's observed standard deviation. We have not yet corrected for the biases that performance fees and smoothing introduce. Not surprisingly, these distortions imply (implausibly) that investors should require a lower expected return from illiquid equity than from liquid equity.

In column 3, we correct for the effect of performance fees on the observed standard deviation and illiquid equity correlation, using Equations (4) and (5). This correction raises the illiquid equity's required return.

In column 4, we apply Equations (6) and (7) to correct for the smoothing that arises from fair-value pricing, which shows that investors should require a premium to justify the substitution of illiquid equity for liquid equity.

In column 5 we introduce the shadow liquidity asset and adjust the bond component's expected return and standard deviation, as well as its correlation with illiquid equity, using equations (8), (9), and (10). This

EXHIBIT 1

Required Return for Liquid and Illiquid Equities

	1	2	3	4	5
	Liquid Equity	Illiquid Equity Unadjusted	Correct for Fee Asymmetry	Correct for Smoothing	Correct for Liquidity
Required equity return	8.75%	5.31%	5.75%	13.75%	15.50%
Bond return	5.00%	5.00%	5.00%	5.00%	7.00%
SLA return	2.00%	2.00%	2.00%	2.00%	2.00%
Equity standard deviation	20.00%	7.50%	10.00%	30.00%	30.00%
Bond standard deviation	5.00%	5.00%	5.00%	5.00%	7.07%
SLA standard deviation	5.00%	5.00%	5.00%	5.00%	5.00%
Equity/bond correlation	0.5000	0.2500	0.3000	0.5000	0.3536
Risk aversion	1	1	1	1	1
Equity weight	50%	50%	50%	50%	50%
Bond weight	50%	50%	50%	50%	50%
Marginal utility equities	0.04250	0.04656	0.04600	0.04000	0.05750
Marginal utility bonds	0.04250	0.04656	0.04600	0.04000	0.05750
Derivative difference	0.0000	0.0000	0.0000	0.0000	0.0000

final step gives the total expected return required of illiquid equity, taking into account the distortions introduced by performance fees and smoothing and the opportunity cost of forgoing liquidity.

The required illiquidity premium equals the difference between the required return in column 5 and the required return in column 4, which in our example equals 1.75%. This required illiquidity premium is slightly less than the shadow liquidity asset's 2% expected return, because the shadow asset introduces risk as well as incremental expected return to the portfolio.

Suppose that illiquid equity has an expected return of 14.70% instead of 15.50%, offering an illiquidity premium of only 0.95%, compared to the required illiquidity premium of 1.75%. In this case, the optimal allocation to illiquid equity would fall from 50% to 45%, assuming all other assumptions remain unchanged. Alternatively, we could maintain our original 50% allocation to illiquid equity and solve for the shadow asset's required return, given an expected return of 14.70% for illiquid equity. In this case, we would require an expected return of only 1.20%, instead of 2%, from the shadow asset.¹⁰

It is important to note that even though there is a single market price for liquidity investors do not benefit equally from liquidity. Two investors with identical expectations and preferences, but different in the extent to which they benefit from liquidity, should not hold the same portfolio, just as investors with higher tax rates should be more inclined than investors with lower tax rates to hold tax-favored assets, such as municipal bonds.

PERFORMANCE FEES AND MULTIPLE FUNDS

We have already shown that performance fees cause the observed standard deviation of a fund to understate its risk. Performance fees also reduce the expected return of a group of funds that charge performance fees beyond the average reduction in the individual funds' expected returns. Consider, for example, a fund that charges a base fee of 2% and a performance fee of 20%. A fund that delivers a 10% return in excess of the benchmark on a \$100 million portfolio collects a \$2 million base fee ($2\% \times 100,000,000$) and a \$1.6 million performance fee ($20\% \times (10,000,000 - 2,000,000)$), for a total fee of \$3.6 million. The investor's net-of-fees return is 6.4%.

Now suppose an investor hires two funds that each charge a base fee of 2% and a performance fee of 20%.

Assume that these funds both have expected returns of 10% in excess of their benchmarks and that the investor allocates the same amount of capital to each. Each fund's expected fee is 3.6% ($2\% + 20\% \times (10\% - 2\%)$). The investor might expect an aggregate net return from these two funds of 6.4%. This will happen, however, only if both funds' returns exceed the base fee. If one fund produces an excess return of 30% and the other a -10% excess return, the investor pays an average fee of 4.8%, not 3.6%, and gets an average return of 5.2%, not 6.4%, even though the funds still have an average excess return of 10%.¹¹ Exhibit 2 summarizes these results.

This result is specific to this example's assumptions. Nevertheless, a Monte Carlo simulation easily determines the typical reduction in expected return. Consider an investment in 10 funds, each with an expected excess return of 7%, a standard deviation of 15%, and a correlation of 0% with the other funds. Assume that the risk-free rate (benchmark) is 4%. The reduction to the collective expected fund return is about 0.7%. If the funds' correlations were higher, the reduction would be smaller and vice versa. This reduction in the collective return is a hidden fee that arises from the fact that investors pay for outperformance but are not reimbursed for underperformance. Exhibit 3 shows how the values derived in this example would differ if an investor substituted multiple funds for a multi-strategy fund.

EXHIBIT 2 Multi-Fund Return Impact

	Excess Return	Manager Fee	Return to Investor
Fund 1	10.0%	3.6%	6.4%
Fund 2	10.0%	3.6%	6.4%
Average	10.0%	3.6%	6.4%
Fund 1	30.0%	7.6%	22.4%
Fund 2	-10.0%	2.0%	-12.0%
Average	10.0%	4.8%	5.2%
Impact			1.2%

EXHIBIT 3 Multi-Strategy Fund vs. Multiple Funds

	Multi-Strategy Fund	Multiple Funds
Required liquidity premium	1.75%	2.45%
Optimal allocation	50.00%	46.00%
Required shadow asset return	2.00%	1.30%

In principle, this effect could be somewhat muted, because most performance-fee arrangements include claw back provisions that require funds to offset prior losses before collecting performance fees. In most cases, though, managers either terminate underperforming funds or reset performance fees without reimbursing investors for losses.

CASE STUDY

We now present a case study in which we apply our approach to a representative institutional portfolio, as shown in Exhibit 4. Our objective is to compare the optimal allocations that account for liquidity with those that ignore it.

These allocations are optimal for a mean–variance investor whose risk aversion coefficient equals 2, given the assumptions for expected returns, standard deviations, and correlations shown in Exhibit 5. We adjust standard deviations to account for the effect of performance fees and smoothing, as described earlier. These weights do not yet take liquidity into account.

EXHIBIT 4

Optimal Portfolio Weights (Ignoring Liquidity)

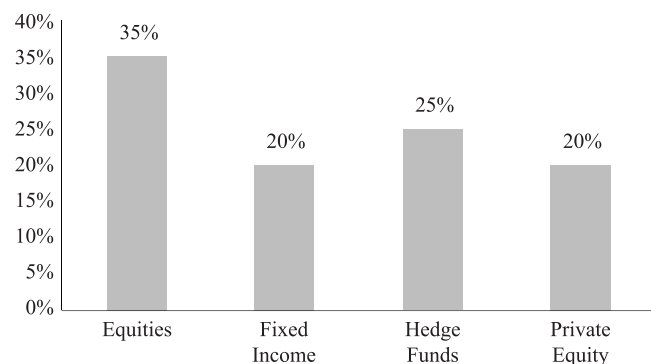


EXHIBIT 5

Asset Class Expected Returns, Risk, and Correlations

Asset Class	Expected Return (%)	Standard Deviation (%)	Correlations			
			Equities	Fixed Income	Hedge Funds	Private Equity
Equities	9.4	15.0	1.00			
Fixed Income	4.0	8.0	0.25	1.00		
Hedge Funds	7.0	15.0	0.25	0.25	1.00	
Private Equity	15.8	30.0	0.75	0.00	0.25	1.00

We estimate the expected return and risk arising from three uses of liquidity: market timing, portfolio rebalancing, and funding capital calls. Investors typically estimate the expected return and risk of explicit assets from some combination of historical data and theoretical pricing models. However, there are no data for shadow allocations, nor are there any theories on which to rely, so we use simulation to estimate the shadow allocations' return and risk.

Market Timing

We assume that skillful market timing produces an excess return equal to 0.40%, with excess risk of 0.80%. We derive these values by simulating a market-timing strategy's performance. Because liquidity serves to improve expected utility we attach these values as a shadow asset to the liquid assets.

Rebalancing

We simulate 10,000 five-year paths given assumptions about the portfolio allocation as well as the expected return, risk and correlations for each asset class.¹² We consider two scenarios: one in which we rebalance the portfolio to its target weights annually (at a cost of 0.2%) and one in which we do not rebalance the weights. We compute the illiquidity penalty as the difference between the portfolios' ending wealth distributions' certainty equivalents in the two scenarios. Because liquidity serves to restore expected utility and not to improve it, we attach this difference in certainty equivalents as a shadow liability to the illiquid assets.

Capital Calls

We assume a 10% probability the investor will need to raise cash to meet capital calls in a given month.

When a call arises, its size is drawn from a known probability distribution.¹³ As long as cumulative excess capital calls are below 15% of initial portfolio value, we draw on the liquid assets (proportionally) to meet capital calls. When cumulative excess capital calls exceed 15% of initial portfolio value, we begin borrowing to meet capital calls, at a rate of 5%. We do not allow cumulative excess capital calls to exceed 20% of initial portfolio value. Using the same 10,000 paths from the rebalancing simulation, we take the difference between the certainty equivalents of two scenarios: one in which we remove capital calls proportionally from all assets and rebalance annually, and another in which we remove capital calls only from the liquid assets and do not rebalance. The incremental sub-optimality associated with capital calls is equal to the sub-optimality from this simulation minus the sub-optimality from the simulation that accounts for rebalancing, but not for capital calls.¹⁴ We deploy liquidity to restore optimality, so we attach a shadow liability to the illiquid assets that we cannot rebalance. We also attach a shadow liability to the illiquid assets to account for the cost and uncertainty of borrowing.¹⁵ Exhibit 6 summarizes these shadow allocations.

Next, as shown in Exhibit 7, we augment Exhibit 5 to account for the shadow asset and liability.

Finally, as shown in Exhibit 8, we re-optimize the portfolio to account for the effect of the shadow asset and liability. The bars to the left show the optimal allocations, assuming we ignore liquidity. They are the same as those shown in Exhibit 4. The bars in the center show the optimal allocations, taking liquidity into account, but assuming there is a single multi-strategy illiquid fund. With this assumption, the hedge fund and private equity allocations fall by 44% and 30%, respectively. The bars to the right assume multiple illiquid funds, which further reduces the allocation to both illiquid asset classes.¹⁶

This case study, though rooted in the real world, represents a first pass at implementing our model, so it overlooks a variety of complexities. For example, we assume that the shadow asset and liability are uncorrelated with the portfolio's explicit assets. We also assume that portfolio returns are serially independent and drawn from the same distribution. We model both absolute and partial illiquidity as though they are constant across time and assets. We assume that the effects of liquidity are additive. Finally, we assume that the returns of the shadow

EXHIBIT 6

Liquidity Benefits and Illiquidity Penalties

	Return (bps)	Risk (bps)	Attached to:
Benefits			
Market timing	40	80	Liquid assets
Total shadow asset	40	80	Liquid assets
Penalties			
Sub-optimality cost from asset drift	46	0	Illiquid assets
Sub-optimality cost from capital calls	3	0	Illiquid assets
Borrowing cost from capital calls	20	16	Illiquid assets
Total shadow liability	69	16	Illiquid assets

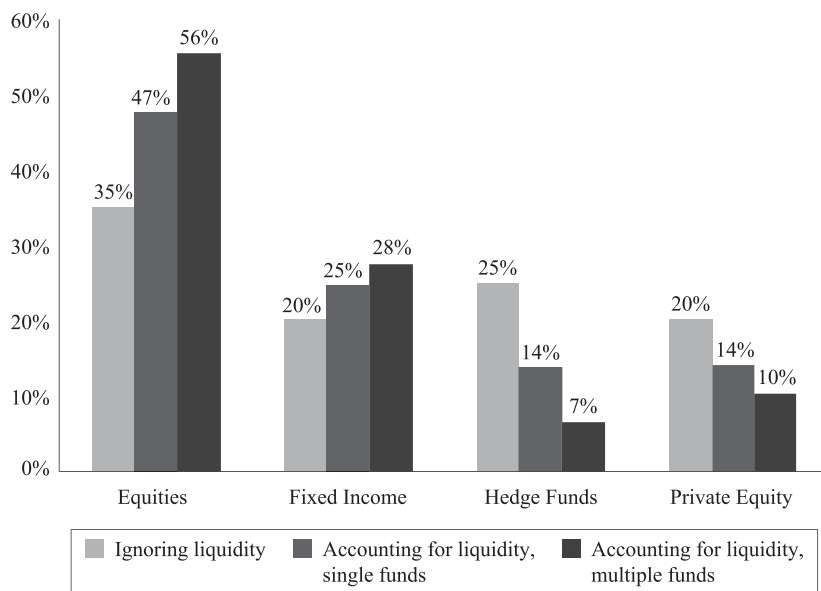
EXHIBIT 7

Asset Class Expected Returns, Risk, and Correlations

Asset Class	Expected Return (%)	Standard Deviation (%)	Correlations					
			Equities	Fixed Income	Hedge Funds	Private Equity	Shadow Asset	Shadow Liability
Equities	9.4	15.0	1.00					
Fixed Income	4.0	8.0	0.25	1.00				
Hedge Funds	7.0	15.0	0.25	0.25	1.00			
Private Equity	15.8	30.0	0.75	0.00	0.25	1.00		
Shadow Asset	0.40	0.80	0.00	0.00	0.00	0.00	1.00	
Shadow Liability	-0.69	0.16	0.00	0.00	0.00	0.00	0.00	1.00

EXHIBIT 8

Optimal Portfolio Weights (with and without liquidity)



Notes: We assume a risk aversion of 2. For the multiple funds scenario, we assume that hedge funds and private equity each consist of ten managers, and reduce both asset classes expected return by 0.70% to reflect the multi-fund return impact of performance fees.

allocations are normally distributed, though in some cases it may be more appropriate to model them as options. We could enrich our analysis by relaxing these assumptions.

SUMMARY

Investors have long struggled to account for liquidity in portfolio choice. Although other researchers have put forth clever models and heuristics for incorporating liquidity into portfolio choice, these approaches are either incomplete or fail to place liquidity in a common unit with return and risk. We introduce an analytical construct that treats liquidity as shadow allocations. If liquidity is deployed to raise expected utility, we attach a shadow asset to tradable assets to capture this incremental benefit. Therefore, if any fraction of a portfolio is immobile, investors bear an illiquidity cost which they should take into account when forming portfolios.

Our model improves upon heuristic approaches for incorporating liquidity into portfolio choice in four fundamental ways. First, it mirrors what actually occurs within a portfolio. Second, it allows investors to map liquidity onto units of expected return and risk, enabling them to analyze liquidity in the same context as other

portfolio decisions. Third, it allows investors to treat absolute illiquidity and partial illiquidity within a single, unifying framework. Fourth, it accounts for the fact that liquidity serves not only to meet demands for capital, but also to exploit trading opportunities. If any fraction of a portfolio is immobile, investors bear an illiquidity cost; they should take that cost into account as they form portfolios.

ENDNOTES

We thank Antti Ilmanen, Hans Luedemann, Roger Stein, and participants at JOIM, the European Quantitative Forum, and the MIT Sloan School Finance Ph.D. Seminar for helpful comments.

¹Investors can determine the appropriate rebalancing schedule by comparing the cost of a suboptimal portfolio with the cost of restoring the optimal asset weights. Kritzman et al. [2009] provide a dynamic programming solution to the optimal rebalancing problem, as well as a solution based on a quadratic heuristic, developed by Erik van Dijk and Harry Markowitz, to overcome the curse of dimensionality.

²The certainty equivalent return for a given portfolio is the amount of risk free return required such that a given investor is indifferent between receiving that return and holding a particular risky portfolio.

³We illustrate our approach with mean-variance optimization, but it can be applied to any portfolio formation process, including full-scale optimization, multi-period models, and even heuristic approaches.

⁴See, for example, Markowitz [1952].

⁵See, for example, Sharpe [1987].

⁶We simulate 1,000 years of monthly fund returns from a normal distribution with an annualized mean of 8% and an annualized standard deviation of 8%. We assume an annual base fee of 2% and an additional annual hurdle rate of 3.5% before performance fees. We record a 20% annual performance fee on an accrual basis each month. If cumulative profit (net of the hurdle) turns negative for the year, we assess a negative performance fee on each month's returns, maintaining a minimum annual fee of zero.

⁷Many studies have found evidence of positive serial correlation in private market investment returns. See Giliberto [2003] as an example.

⁸We assume that the shadow asset is uncorrelated with both stocks and bonds, an assumption we can relax.

⁹For convenience we assume that liquid asset trades are free of charge. We relax this assumption in the case study presented in the following section.

¹⁰We present this analytical construct for expository purposes. In practice, one could simply introduce the shadow asset as an overlay that does not require capital and constrain its weight to equal the sum of the liquid assets.

¹¹This difference is equivalent to the difference in value between a portfolio of options and an option on a portfolio.

¹²To compute sub-optimality cost we assume the initial portfolio weights are optimal. But accounting for this cost renders the initial weights sub-optimal. We therefore need to recalculate it iteratively to arrive at a more precise estimate.

¹³We assume a 10% probability the investor will need to raise cash to meet capital calls in a given month. When a call arises, there is a 70% chance its size is equal to 1% of the starting portfolio value, a 20% chance its size is 2% of the starting portfolio value, and a 10% chance its size is 3% of the starting portfolio value.

¹⁴For reasonable assumptions about the size and likelihood of capital calls, the sub-optimality caused by capital calls tends to be much smaller than the sub-optimality caused by drifting asset prices.

¹⁵The borrowing cost we estimate in the simulation assumes the asset allocation shown in Exhibit 4; therefore, we express this cost as a percentage of the allocation to illiquid assets.

¹⁶In this example, we have in mind median performing hedge fund and private equity managers. We understand that managers who choose to read our article are relatively sophisticated and would most likely perform in the top quartile or better. Hence, we are willing to acknowledge that their existing allocations could be optimal, even taking liquidity into account.

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